A CLOSER LOOK AT INFINITESIMALS

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> "Calculus required continuity, and continuity was supposed to require the infinitely little; but nobody could discover what the infinitely little might be."
>
> Bertrand Russell

In this writeup, we examine the concept of infinitesimals in the works of both Newton and Leibniz and expand on why their initial concept of the infinitesimal failed to address both mathematical and philosophical problems. Historically, this led to the development of real analysis and with theory of limits being rigorously introduced by Cauchy and Weierstrass in the 19th century, the use of infinitesimals in math was abandoned and made obsolete. Despite this new and rigorous foundation, much of the new theory was (and still is) rather unintuitive in the physical sense. However, we argue that it is not necessary to completely abandon infinitesimals, but rather under a different framework of mathematics, we can make the infinitesimal mathematically rigorous.

1. Do Infinitesimals Exist?

Before we can discuss the mathematical and philosophical problems of the infinitesimal, we must first define it. The original definition of the infinitesimal utilised by Leibniz was "a quantity that does not coincide with zero, but it is smaller than any finite quantity". Newton, on the other hand, defined infinitesimals to be "evanescent quantities" in order to capture the idea of instantaneous velocity [Bel19]. However in both definitions, the core idea is that an infinitesimal represents a quantity that is small enough such that it is negligible with respect to other quantities; yet the sum of these negligible quantities comprise whole entities. That is, infinitesimals are nonzero objects that are indistinguishable from zero. This concept has always been a philosophical issue and we first elaborate on these issues that follow from classical logic.

First, we consider the continuum, which is a closely related but a more fundamental and intuitive idea, as we perceive time and space as continuous entities. A continuum has the property that it is always divisible. That is, we can always partition a continuum into smaller continua and this process will never terminate. Doing so implies that in a continuum, there is no such thing as an "indivisible quantity". Consequently, infinitesimals were conceived to be the parts comprising a continuum just as a discrete object consists of individual units. Thus, it follows that infinitesimals themselves are continua and therefore are divisible, meaning that they cannot be points as points are indivisible. That is, no line consists of points and no points can be a part of a line and a similar statement holds for infinitesimals. Due to the divisibility of continua, we get that infinitesimals are composed of smaller infinitesimals and therefore their magnitudes must be variable. Expanding on this idea, we conclude that infinitesimals can only possess potential magnitudes which can generate magnitudes, but they will not have magnitudes themselves. However, this would imply that infinitesimals are not actually quantities at all. Although Leibniz used and believed that infinitesimals were variables, we have that infinitesimals cannot actually be a quantities themselves, conflicting with his definition.

Similarly, Newton's definition of the infinitesimal had its own issues. Newton's notion of an evanescent quantity is slightly ill-defined and faces similar problems as Zeno's arrow paradox [Dow19]. The arrow paradox is as follows: Suppose an arrow is show towards a target. We observe that the arrow is traveling through space and time. However, in an infinitesimal amount of time, how much space does this arrow occupy? If the arrow does not grow larger in the infinitesimal amount of time, then we get an arrow which will not travel at all, but if it does grow larger, then there is a different arrow, and how does our original arrow transform into this slightly larger arrow? We know that these instant moments are nonzero amounts of time but how can motion occur during any specific evanescent moment? The philosopher George Berkeley, in his work *The Analyst* (1734), also attacked Newton's definitions [Bel19]. Berkeley wrote:

> "...And what are these fluxions? The velocities of evanescent increments? And what are these same evanescent increments? They are neither finite quantities nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?"

Berkeley is claiming that Newton first assumes that there is a slight change in a system but then acts as if there was no change at all whenever convenient.

In the math world, the idea of the infinitesimal number was also shown to conflict with the Archimedean principle of the real numbers. That is, the infinitesimal numbers had the characterizing property that if you add an infinitesimal number to itself an arbitrary number of times, the sum will remain less than any finite number. One attempted construction to justify this property, which also ended up being the typical use of infinitesimals, was provided by Johann Bernoulli in his letter to Leibniz in 1698. He stated that "inasmuch as the number of terms in nature is infinite, the infinitesimal exists ipso facto" [Bel19]. He goes onto argue for the existence of infinitesimals by considering the infinite harmonic sequence $\{1/n : n \in \mathbb{Z}^+\}$. For example, if we consider the first 10 terms, then we have that $1/10$ exists. Similarly, if we consider the first 100, then we have that $1/100$ exists and so if we do consider the infinite sequence, we should get that the existence of a term $1/x$ for some $x \in \mathbb{Z}^+$ that is nonzero but indistinguishable from zero. However, Cantor refutes this construction, as we will see later.

Leibniz, in an effort to defend his work, ultimately concludes that his construction contradicts itself in a philosophical manner [Jes08]. He writes in his letter to the philosopher Des Bosses:

> "...philosophically speaking, I no more admit magnitudes infinitely small than infinitely great...I take both for mental fictions, as more convenient ways of speaking, and adapted to calculation, just like imaginary roots are in algebra."

At the time, Leibniz retreated to considering the infinitesimal as a fictional, but useful tool that churned out correct results and clean arguments, while promoting insight and advancement in science and engineering.

However, Cantor reiterates that this does not justify the use of infinitesimals since mathematics should be "bound only by the self-evident concern that its concepts be both internally without contradiction and stand in definite relations, organized by means of definitions, to previously formed, already existing and proven concepts" [Hal84]. This is a reasonable justification which allows engineers and physicists to continue utilizing these useful, but flawed ideas while emphasizing that mathematicians and philosophers should still be concerned and that they must continue to work on providing a better theory. To emphasise this, Cantor set out to demonstrate that the concept of the infinitesimals were inconsistent. From his statement, he reasoned that it would suffice to show that infinitesimals may be justifiably abandoned if they were provably inconsistent. Cantor, in his letter to Weierstrass, writes

> "Linear number magnitudes ζ different from zero (i.e. shortly put, such number magnitudes as can be represented by bounded, straight, continuous segments) which would be less than any ever so small finite number magnitude do not exist, i.e. they contradict the concept of a linear number magnitude...I begin from the supposition of a linear magnitude ζ which is so small that its product by n, $n \cdot \zeta$, for every finite whole number n however great is smaller than unity, and then prove, from the concept of a linear magnitude...that then $\zeta \cdot \nu$ is less than every finite magnitude however small, where ν is an arbitrarily great transfinite ordinal number from any arbitrarily high number class. But this means that ζ cannot be made finite by any actually infinite multiplication of any power, and hence surely cannot be made an element of finite magnitudes. But then the supposition made contradicts the concept of a linear magnitude...Hence, there remains no alternative but to drop the supposition that there is a magnitude ζ which is smaller than $1/n$ for every finite whole number n." [Moo02]

In his argument, Cantor showed that these infinitesimals cannot be the multiplicative inverses of his transfinite numbers, as proposed by the argument by Bernoulli. Moreover, Cantor successfully showed that the characterizing property of nonzero infinitesimals, which again is that they are smaller than any finite number magnitudes, cannot coincide with the Archimedean property of the real numbers since they cannot be made finite by the multiplication of any number, including a transfinite number. Although Cantor is correct in his argument, his conclusion that this argument banishes the existence of infinitesimals is erroneous, as Hilbert eventually proves the consistency of non-Archimedean geometries and number systems. That is, Cantor's proof only showed that infinitesimals cannot exist in an Archimedean field. It should be noted that Cantor did not believe in the existence of non-Archimedean systems, which would explain the jump in his conclusion.

2. Recovering the Infinitesimal

Despite the criticisms of the original foundations of calculus, it is possible to recover the desired notion of the infinitesimal quantity (as described by Leibniz and Newton) by working in a non-Archimedean setting or by adopting an intuitionist[1] approach to mathematics. We will provide a mathematical framework in which infinitesimals can be used appropriately and that our results will still coincide with the typical calculus results obtained by limit theory.

So let us first consider a (real) line consisting of points corresponding to the real numbers $\mathbb{R}$ and an "infinitesimal" interval on either side of every such point. Here, we mean an infinitesimal to be a quantity that does not satisfy the Archimedean principle. Furthermore, we also impose the property for this line that no two real numbers, or points that is, can differ by an infinitesimal interval since the difference of any two real numbers is a finite real number, and we would like to preserve the characterisation of the infinitesimal. Throughout the rest of this paper, we will denote the real line with the infinitesimals as $\mathcal{R}$. Treating the infinitesimal as a non-Archimedean quantity seems to resolve the issue of Zeno's paradox, as W.I. McLaughlin explains that "a trajectory and its associated time interval are in fact densely packed with infinitesimal regions" such that none of the motions are taking place at the points (the real numbers) but rather during the infinitesimal intervals of time [Art]. Moreover, this also coincides with our intuition of an actual line, as the line is no longer just an object consisting of discrete points, but rather a continuous entity now that we have some sort of ether in the form of the infinitesimals that makes it a continuous line.

One of the most important properties that we wanted from an infinitesimal was that it was a nonzero object that was indistinguishable from zero. Using ring theory, we can begin to mathematically formalise infinitesimals: Let $D_n = \{d \in \mathcal{R} : d^{n+1} = 0\}$ be the set of nilpotent "infinitesimals" of index $n+1$ (we will justify the use of calling these elements infinitesimals.) We propose the following axioms:[2]

- **Axiom 1:** Any function $f : D_n \longrightarrow \mathcal{R}$ has the form $f(d) = a_0 + a_1 d + a_2 d^2 + \cdots + a_n d^n$ where each a_i is uniquely determined.
- **Axiom 2:** If $x \in \mathcal{R}$ such that $x \neq 0$, then x is invertible in $\mathcal{R}$.

Given this, we define the (first) derivative. Fixing $x \in \mathcal{R}$, consider the function $g : D_1 \longrightarrow \mathcal{R}$ given by $g(d) = f(x+d)$. By the first axiom (specifying it for $n = 1$), there exists a unique $b \in \mathcal{R}$ such that

$$g(d) = g(0) + d \cdot b, \quad d \in D_1.$$

That is, we have that

$$f(x+d) = f(x) + d \cdot b, \quad d \in D_1.$$

We denote the element b by $f'(x)$ since b is a variable dependent on x. That is $f'(x)$, which we call the first derivative of f, is the function satisfying the equation

$$f(x+d) = f(x) + d \cdot f'(x), \quad d \in D_1.$$

Since $f'(x)$ is defined on all of $\mathcal{R}$, we have that $f' : \mathcal{R} \longrightarrow \mathcal{R}$. Furthermore, we can repeat this process to obtain n-th order derivatives but when we do so, we consider not just D_1, but increase our index by 1 after each step until we get to D_n. This will demonstrated later in the proof of Taylor's theorem.

[1] Intutionists do not believe in the law of excluded middle, a law which much of mathematics and logic uses and have used since Euclid. To an intuitionist, you can only claim a statement is true if you can provide a constructive proof.

[2] We follow the exposition given by Kock and Shulman (See References).

The second axiom gives us a field structure on $\mathcal{R}$ despite the nilpotency of the elements of $D_n \subseteq \mathcal{R}$. This is because if we take the contrapositive, which states that if x is not invertible, then it is not nonzero and all we can conclude about the nilpotent elements is that they are not nonzero. Thus, the infinitesimals are not necessarily zero, but we cannot actually distinguish any particular infinitesimal from zero either. Remarkably, this coincides with the original idea that an infinitesimal is a quantity that is too small to have a (finite) specific value.

At a first glance, it may seem that these two axioms contradict each other and so we will now show that they are consistent with each other. We do this by showing that $D_n \neq \{0\}$ for any n. It suffices to show that it is true for D_1. Actually, for the remainder of this paper, we will primarily focus on D_1 since working with D_1 provides the foundation for D_n for $n \geq 1$ while capturing all of the main ideas necessary to discuss differential calculus. That is, the following ideas can easily generalized to any $n \geq 1$.

We begin the argument by first giving a lemma which states that although we do not have division of infinitesimals, we do have a similar type of cancellation property for infinitesimals.

Lemma 2.1 (Cancellation Lemma)**.** *Let $a, b \in \mathcal{R}$. If $ad = bd$ for all $d \in D_1 \subseteq \mathcal{R}$, then $a = b$.*

Proof of Lemma. Consider the function $f(d) = ad = bd$. Since $f(d)$ can be expressed as

$$f(d) = f(0) + cd$$

for some unique $c \in \mathcal{R}$, we conclude that $a = c = b$. □

Theorem 2.2. *The set of nilpotent infinitesimals is nonzero.*

Proof. Again it suffices to show the statement is true for $n = 1$. Suppose that $D_1 = \{0\}$. Then consider two functions $f(d) = a + bd = a$ and $g(d) = a + b'd = a$. That is, we have that $f \equiv g$ and so $b = b'$ for all $b, b' \in \mathcal{R}$ by the uniqueness of Axiom 1. However, this is not true since $\mathcal{R} \neq \{0\}$. Therefore, we conclude that $D_1 \neq \{0\}$. □

The result above gives us that the two axioms are not in conflict with each other. Note that the first axiom guarantees that the first n derivatives of any function exist. Moreover, if we write the set of all nilpotent infinitesimals in $\mathcal{R}$ as

$$D_\infty = \{d \in \mathcal{R} : \exists n \in \mathbb{N} \text{ such that } d^n = 0\} = \bigcup_{n \geq 1} D_n,$$

it follows that every function $f : D_\infty \longrightarrow \mathcal{R}$ is analytic on $\mathcal{R}$, where the convergence of the power series representation is guaranteed by the nilpotency of d.

Given these properties, we can now emphasize that these nilpotent elements capture the idea of an "infinitesimal," since these are nonzero quantities that have the property that when we multiply an infinitesimal with any real number, then we have an infinitesimal again and we cannot distinguish them from zero. To see the first property, let $x \in \mathcal{R}$ and let $d \in D_1$. Then we have that since $d^2 = 0$ that

$$(xd)^2 = x^2 d^2 = 0$$

and so $xd \in D_1$ as well. Thus, we have a system where Cantor's argument against infinitesimals no longer applies while preserving the original idea of the infinitesimal. However, it should be noted that for any finite n, we do not have that D_n is an ideal since D_n is not closed under addition of infinitesimals. For example, if $d_1, d_2 \in D_1$, then

$$(d_1 + d_2)^2 = d_1^2 + 2d_1d_2 + d_2^2 = 2d_1d_2 \neq 0.$$

If we did take the equation to be zero, then we would have to conclude by **Lemma 2.1** that $D_1 = \{0\}$, which we already have shown is not true from **Theorem 2.2**. However, we do have that since D_∞ is closed under addition, it forms an ideal in $\mathcal{R}$.

Now that we have a notion of the infinitesimal, and we had already defined the derivative from the axioms, we are now able to obtain properties about the derivative, which fortunately coincide with the limit definition of the derivative.

Theorem 2.3. *Let $f, g : U \longrightarrow \mathcal{R}$ be functions and let $r \in \mathcal{R}$ where $U \subseteq \mathcal{R}$ such that $U = U' := \{x \in U : x + d \in U,\ \forall d \in D_1\}$, then*

(1) $(f + g)' = f' + g'$
(2) $(r \cdot f)' = r \cdot f'$
(3) $(fg)' = f'g + fg'$
(4) $\mathrm{id}'_{\mathcal{R}} = 1$
(5) $r' = 0$ *for any* $r \in \mathcal{R}$.

If $f : U \longrightarrow V$ and $g : V \longrightarrow U$, where $U, V \subseteq \mathcal{R}$ are closed under under the addition of infinitesimals, then we also have

$$(f \circ g)' = \big(f' \circ g\big) \cdot g'.$$

We do not prove these properties, since they follow immediately from the axioms and the cancellation lemma. While the first axiom does provide us with the existence of the first n-derivatives of any function $f : \mathcal{R} \longrightarrow \mathcal{R}$, we also have Taylor's theorem from differential calculus.

Theorem 2.4. *Let $f : \mathcal{R} \longrightarrow \mathcal{R}$ and let $x \in \mathcal{R}$. Then for all $d \in D_k$, we have that*

$$f(x + d) = f(x) + d \cdot f'(x) + \frac{d^2}{2!} f''(x) + \cdots + \frac{d^k}{k!} f^{(k)}(x).$$

The result also holds for any neighbourhood U around x (in case f is not defined on all of $\mathcal{R}$.)

Proof. We will show the result for $k = 2$. From **Axiom 1**, we get that

$$f(x + d) = f(x) + b_1 d + b_2 d^2$$

for some $b_1, b_2 \in \mathcal{R}$. Specializing d to be nilsquare, we get that $b_1 = f'(x)$. Now if we let $d_1, d_2 \in D_2$ be distinct, then we have

$$\begin{aligned} f(x + d_1 + d_2) &= f(x + d_1) + f'(x + d_1)d_2 \\ &= f(x) + f'(x)d_1 + f'(x)d_2 + f''(x)d_1d_2 \\ &= f(x) + f'(x)(d_1 + d_2) + \frac{1}{2} f''(x)(d_1 + d_2)^2 \end{aligned}$$

and specializing $d = d_1 + d_2$, then we get that $b_2 = \dfrac{1}{2!} f''(x)$. □

While we do have that in the intuitionist approach to math that every function is continuous and differentiable, it lacks one of the most fundamental theorems of continuous functions in classical logic: the intermediate value theorem. This is because the theorem only provides the existence of an intermediate value but does not construct it. However, we have an alternative to this theorem but it requires integral calculus. We can try to construct the theory of integration by creating an axiom that guarantees the existence of primitive functions just as we had an axiom providing the existence of the derivative function. However, before we can formulate such an axiom, we need to discuss the order properties of the line $\mathcal{R}$.

We know that $\mathcal{R}$ has a field structure and so give $\mathcal{R}$ the preordering relation $\leq$ such that it satisfies the canonical ordering on $\mathbb{R}$ when we consider only real numbers. Consequently, we have the transitive property, reflexive property as well as the following properties:

(1) $x \leq y \Longrightarrow x + z \leq y + z$
(2) $x \leq y, t \geq 0 \Longrightarrow xt \leq yt$
(3) $0 \leq 1$
(4) d nilpotent $\Longrightarrow 0 \leq d$ and $d \leq 0$.

Again, the lack of antisymmetry (4) emphasizes the notion that a nonzero infinitesimal cannot be distinguished from zero but it is less than any finite quantity with respect to this preorder.

We define an interval $[a, b]$ to be, as expected, to be the set

$$[a, b] = \{x \in \mathcal{R} : a \leq x \leq b\}.$$

Note that the interval $[a, b]$ is indistinguishable from intervals of the form $[a, b + d]$, $[a + d, b]$ or $[a + d_1, b + d_2]$ for $d, d_1, d_2 \in D_n$. Moreover, from these properties, we get that any interval $[a, b]$ is also convex since for any $x, y \in [a, b]$, we have that $x + t(y - x) \in [a, b]$ for $0 \leq t \leq 1$.

We can now readily state our axiom of integration:

Axiom 3: For any $f : [0, 1] \longrightarrow \mathcal{R}$, there exists a unique $g : [0, 1] \to \mathcal{R}$ such that $g' \equiv f$ and $g(0) = 0$.

From this axiom, we can define the (definite) integral of a function f on the interval $[0, 1]$ as

$$\int_0^1 f(t)dt =: g(1) \quad (= g(1) - g(0))$$

and so we also get that for any function $h : [0, 1] \to \mathcal{R}$, we have that

$$\int_0^1 h'(t)dt = h(1) - h(0).$$

That is, we have that the integral coincides with the notion of the antiderivative. This gives us all of our desired properties of the Riemann integral such as additivity, linearity, etc. From this, we get the following result:

Theorem 2.5 (Hadamard's Result)**.** *Let $a, b \in \mathcal{R}$ and $f : [a, b] \to \mathcal{R}$ be given. For any $x, y \in \mathcal{R}$, we have*

$$f(y) - f(x) = (y - x) \int_0^1 f'(x + t(y - x))dt.$$

Proof. Define a map $\phi : [0,1] \to [a,b]$ by $t \longmapsto x + t(y-x)$ and so we get that $\phi' \equiv y - x$. Thus, we have that

$$\begin{aligned} f(y) - f(x) = f(\phi(1)) - f(\phi(0)) &= \int_0^1 (f \circ \phi)'(t) dt \\ &= \int_0^1 f'(\phi(t)) \cdot \phi'(t) dt \qquad \text{by the chain rule} \\ &= \int_0^1 (y-x) f'(\phi(t)) dt \end{aligned}$$

and the linearity property of our notion of the integral gives us our desired result. □

We remark that it is possible to generalise both the differential and integral calculus results obtained by infinitesimals to higher dimensional calculus and differential geometry. Going into the construction and ideas for multivariable calculus is rather long and as a result are excluded from this paper. However, results in differential multivariable calculus are rather easy to generalize and the theory can be captured relatively quickly and easily. On the other hand, integral multivariable calculus depends on the very rich and beautiful theory and construction of exterior differential forms.

However, the more important idea is that in the right framework, infinitesimal calculus is possible and it is not necessary to abandon infinitesimals. We had seen that the theory of infinitesimals considered by Leibniz and Newton had suffered mainly because they were not in the appropriate framework. More importantly, the mathematical theory had not yet been developed enough to provide a rigorous context for infinitesimals. We saw that with the development of both modern algebra and differential geometry, and with the development of category theory and intuitionism, we are able to accommodate the notion of an infinitesimal. Moreover, the use of infinitesimals continue to provide simple arguments, positively advertising for the rich intuitionistic approach to mathematics.

References

[Art] Richard T. W. Arthur, *Actual infinitesimals in leibniz's early thought.*

[Bel19] John L. Bell, *Continuity and Infinitesimals*, The Stanford Encyclopedia of Philosophy (Edward N. Zalta, ed.), Metaphysics Research Lab, Stanford University, 2019 ed., 2019.

[Dow19] Brad Dowden, *Zeno's Paradoxes*, Internet encyclopedia of philosophy, 2019 ed., 2019.

[Hal84] Michael Hallett, *Cantorian set theory and limitation of size*, Oxford, England: Clarendon Press, 1984.

[Jes08] Douglas Jesseph, *Truth in fiction: Origins and consequences of leibniz?s doctrine of infinitesimal magnitudes*, Infinitesimal Differences: Controversies Between Leibniz and His Contemporaries (Douglas Jesseph and Ursula Goldenbaum, eds.), Walter de Gruyter, 2008.

[Koc06] Anders Kock, *Synthetic differential geometry*, Cambridge University Press, 2006.

[Moo02] Matthew Moore, *A cantorian argument against infinitesimals*, Synthese **133** (2002), 305–330.

[Shu] Michael Shulman, *Chicago pizza-seminar: Synthetic differential geometry.*