Resolutions of Rees Algebras

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"If you want to go fast, go alone. If you want to go far, go together"- an African proverb.

"Repetition is the mother of all learning" - Anton Fonarev

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Table of Contents

Abstract

Resolutions are a topic of interest in modern algebra and are used to study modules. Resolutions are useful since they often encode information about a module, including different invariants such as the projective dimension, regularity, and Hilbert series of a given module. Another object of interest in commutative algebra is the Rees algebra, which captures information about an ideal I and its higher powers. In this thesis, we relate the two topics by looking at the minimal graded free resolution of the Rees algebra of an ideal I and taking degree-d strands of this resolution to give a graded free resolution of I^d. We give a bound on the regularity of I^d through this process. We also provide a detailed example going through the process of trimming the graded free resolution to obtain the value of the regularity from the degree-d strand.

Dedication

To my family and friends.

Introduction

Linear algebra, the study of vector spaces and linear transformations, is of interest due to its applicability to different fields of study. The concepts in linear algebra can be generalized to the theory of modules. For example, finite dimensional vector spaces generalize to finitely generated modules. However, studying the generators of a given module does not provide the full picture of the structure of the module as there are often relations between these generators. Furthermore, there may be relations between the relations, and so on.

To efficiently study the structure of a finitely generated module from its generators, David Hilbert introduced the idea of *syzygies* and associated *free resolutions* to finitely generated modules. Naively, a free resolution is a sequence of modules and maps constructed from the following process: Take a set of generators $\{m_i\}$ for a finitely generated graded module M and map a graded free module $F_0 \to M$ by sending basis elements of F_0 to the generators $\{m_i\}$. Let M' be the kernel of this map, i.e., it is the submodule containing the relations between the generators of M. By Hilbert's Basis Theorem, M' will also be finitely generated and so take a set of generators $\{m_i'\}$ of M' and take a free module F_1 and map its basis elements onto the generators of $M' \subseteq F_0$ and repeat this process.

If we pick out the minimal number of generators necessary at each step in the process of constructing a free resolution, we obtain a *minimal free resolution* of the module. These generators of the kernels of the maps above in the minimal free resolution of M are called the syzygies of M.

The minimal free resolution of a module introduces different invariants of the module, namely the *graded Betti numbers* and the *Castelnuovo-Mumford regularity*, both of which are topics of research in modern mathematics.

In 1997, Swanson [Swa97] provided a linear upper bound for the regularity of powers d of homogenous ideals $I \subseteq A = \Bbbk[x_1, \ldots, x_n]$ in terms of d. Then in 1999, Cutkosky, Herzog, and Trung in [CHT99] and Kodiyalam in [Kod00] extended Swanson's work by proving that the regularity of a homogenous ideal I is eventually equal a linear function of d. The graded Betti numbers for a module (β_{ij}) are another invariant that is related to the Castelnuovo-Mumford regularity and so a complete classification of the graded Betti numbers would be an interesting, albeit hopelessly difficult, problem to attempt. In 2008, Boij and Söderberg published a paper [BS08], titled *Graded Betti numbers of Cohen-Macaulay modules and the multiplicity conjecture*, where they had the idea of classifying the graded Betti numbers up to a rational scalar. That is, instead of attempting to classify the (β_{ij}) directly, they proposed to classify $t \cdot (\beta_{ij})$ for some $t \in \mathbb{Q}$. This idea can be pictured geometrically as a cone in a vector space over the rationals, where the extremal rays of the cone correspond with the *pure resolutions* of the module. The linear combinations of these pure resolutions are called the *Boij-Söderberg decompositions* of the module.

In 2014, Whieldon proved that for homogeneous ideals I generated by forms of the same degree r, that there was a stabilization in the Betti tables for powers of ideals I^d in the sense that for all $n \geq N$, for some large N, that $\beta_{i,j+rn}(I^d) \neq 0 \iff \beta_{i,j+rN}(I^d) \neq 0$. In 2019, Mayes-Tang in [May19] built on Whieldon's result to show that there is a stabilization in the Boij-Söderberg decomposition for I^d for large enough d when I was a homogenous ideal where all of its generators were the same degree.

In this thesis, we attempt to extend these recent results by studying free resolutions of homogeneous ideals of A, and its powers, that are not generated by a single degree. We originally would have also wanted to study the possible stabilization patterns in the Boij-Söderberg decompositions of I^d after an A-*splitting*, which introduces finitely many more variables raised to integral powers, but this was hindered by unforeseen circumstances.

Computations were performed by hand and by Macaulay2.

Chapter 1

Necessary Background

The central objects of study in linear algebra are vector spaces over fields. We can generalize this notion by introducing *modules* over rings. Indeed, as scalars in a vector space come from a field, the scalars of a module come from a ring. Consequently, the reader familiar with linear algebra will have had some experience working with modules, but as fields provide more rigidity than rings in their respective structures, different nuances may arise. For example, not all modules have a basis. This chapter treats the basics of modules, which provide the necessary ideas and tools for this thesis.

1.1 Module Theory

Unless otherwise stated, all rings considered are assumed to be commutative rings with identity. Let A be such a ring.

Definition 1.1.1. An **A-module** $(M, +)$ is an abelian group with an action of A, i.e., a map $A \times M \to M$, denoted $(a, m) \mapsto am$, such that for all $m, n \in M$ and $a, b \in A$, the following axioms are satisfied:

1. $a(m + n) = am + an$
2. $(a + b)m = am + bm$
3. $(ab)m = a(bm)$
4. $1_A m = m$.

Example 1.1.2.
1. If the ring A is a field, then the module M is a vector space over A.
2. Every ring is a module over itself.
3. If G is an abelian group, then G is a $\mathbb{Z}$-module.

Definition 1.1.3. Let M and N be A-modules. An A**-module morphism** (or A-linear map) is a map $f : M \to N$ such that for all $m, n \in M$ and $a \in A$, we have

$$f(m+n) = f(m) + f(n)$$
$$f(am) = af(m).$$

Again, if A is a field, then a module morphism is simply a linear transformation between vector spaces over A.

The set of all A-module morphisms from M to N is denoted as $\text{Hom}_A(M, N)$ (or simply as $\text{Hom}(M, N)$ if there is no ambiguity of the base ring). Note that this set carries a natural A-module structure if we define addition and scalar multiplication by

$$(f+g)(m) = f(m) + g(m)$$
$$(af)(m) = af(m)$$

for all $m \in M$.

Definition 1.1.4. An **isomorphism** of A-modules M and N is a bijective A-module morphism $f : M \to N$. If there is an isomorphism $M \to N$, we say M and N are **isomorphic** and write $M \cong N$.

Definition 1.1.5. A **submodule** N of M is a subgroup with respect to addition of M that is closed under the action by A. That is, for all $a \in A$ and $n, n' \in N$, we have that $n + n' \in N$ and $an \in N$ as well.

Example 1.1.6. If we view the ring A to be a module over itself, all of the submodules of A are precisely the ideals of A.

Example 1.1.7. If $a \in A$ and M is an A-module, then $aM = \{am : m \in M\}$ is a submodule of M. If I is an ideal of A, then IM which the set of all finite linear combinations of $\{am : a \in I, m \in M\}$ is a submodule of M.

Let M be an A-module and let N be a submodule of M. Since M is an abelian group, N is a normal subgroup and so it is possible to define the quotient group M/N. To give this group an A-module structure, we wish to give the canonical projection

$$\pi : M \longrightarrow M/N$$
$$m \longmapsto m + N$$

the structure of an A-module morphism, i.e.

$$a(m+N) = a\pi(m) = \pi(am) = am + N.$$

This leads us to the following definition:

Definition 1.1.8. Let M be an A-module and N be a submodule of M. The **quotient module** M/N is the A-module

$$M/N = \{m + N : m \in M\}$$

with the A-action given by $a(m+N) = am + N$ for all $a \in A$.

Definition 1.1.9. If M and N are A-modules, then the **direct sum** of M and N is the module

$$M \oplus N = \left\{(m,n) : \ m \in M, n \in N\right\}$$

with addition defined componentwise, and scalar multiplication defined $a(m,n) = (am, an)$.

It is quite natural to extend this definition to families of modules. Moreover, similar considerations hold for direct sum of finite set of modules but we must handle infinite sets of modules with more caution.

Definition 1.1.10. Let $\{M_i\}_{i\in I}$ be a family of A-modules. The **direct product** $\prod_i M_i$ is the A-module whose elements are tuples $(m_i)_{i\in I}$ while the **direct sum** $\bigoplus_i M_i$ is the A-module consisting of all tuples $(m_i)_{i\in I}$ such that all but finitely many m_i are zero.

It is clear that the direct sum is a subset of the direct product.

Definition 1.1.11. Let $f : M \to N$ be an A-module morphism. The **kernel** of f is the set

$$\text{ker}(f) = \left\{m \in M : f(m) = 0\right\}.$$

The **image** of f is the set

$$\text{im}(f) = f(M).$$

We note that both the kernel and the image are submodules of M and N, respectively. The **cokernel** of f is the set

$$\text{coker}(f) = N/\text{im}(f)$$

and is a quotient module of N.

Theorem 1.1.12 (Isomorphism theorems for Modules)**.** *Let M be an A-module.*

1. *If $f : M \to N$ is a module morphism, then we have that $M/\ker(f) \cong \mathrm{im}(f)$.*

2. *If $M'' \subseteq M' \subseteq M$ are modules, then*

$$\frac{M/M''}{M'/M''} \cong M/M'.$$

3. *If N and N' are submodules of M, then $N + N'$ is a submodule of M as well and we also have an isomorphism*

$$N/(N \cap N') \cong (N + N')/N'.$$

Proof. The proofs for modules and groups are very similar and are omitted here. See [AM94], pages 18 and 19 for more details. □

Definition 1.1.13. A sequence of A-modules and of A-module morphisms

$$\cdots \xrightarrow{f_{i-1}} M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \cdots$$

is called a **chain complex** (or simply **complex**) if we have that $f_{i+1} \circ f_i = 0$ for all i. A sequence is said to be **exact** at M_i if $\mathrm{im}(f_i) = \ker(f_{i+1})$. A complex that is exact at each M_i is called an **exact sequence**. An exact sequence of the form

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is called a **short exact sequence**.

Depending on context, it may be more convenient to write the complex in the reversed order along with the arrows. That is, we will sometimes write the complex in Definition 1.1.13 as

$$\cdots \xleftarrow{f_{i+1}} M_{i+1} \xleftarrow{f_i} M_i \xleftarrow{f_{i-1}} M_{i-1} \longleftarrow \cdots$$

We will freely reverse the complex without any comment.

Proposition 1.1.14. *Let M, N be A-modules. Then*

1. *$0 \to M \xrightarrow{\phi} N$ is exact if and only if ϕ is injective.*

2. *$M \xrightarrow{\phi} N \to 0$ is exact if and only if ϕ is surjective.*

3. *$0 \to M \xrightarrow{\phi} N \to 0$ is exact if and only if ϕ is an isomorphism.*

4. $0 \to M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \to 0$ *is a short exact sequence if and only if* ϕ *is injective,* ψ *is surjective, and* ψ *induces an isomorphism* $\mathrm{coker}(\phi) \cong M''$.

Definition 1.1.15. Suppose we have a chain complex $C.$ of A-modules

$$C.: \ \cdots \longrightarrow M_{i+1} \xrightarrow{f_{i+1}} M_i \xrightarrow{f_i} M_{i-1} \xrightarrow{f_{i-1}} \cdots$$

Since the composition of two maps is zero, we have

$$0 \subseteq \mathrm{im}(f_{i+1}) \subseteq \ker(f_i) \subseteq M_i, \quad \text{for all } i.$$

The i^{th} **homology module** of $C.$ is defined to be the quotient $H_i(C.) = \ker(f_i)/\mathrm{im}(f_{i-1})$.

Definition 1.1.16. The **annihilator** of an A-module M is

$$\mathrm{Ann}_A(M) = \{a \in A : am = 0, \, \forall m \in M\}.$$

The annihilator of a module is an ideal of the base ring.

Definition 1.1.17. The **Krull dimension of a ring** A is defined to be

$$\dim(A) = \sup\{k : \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_{k-1} \subsetneq \mathfrak{p}_k, \ \ \mathfrak{p}_i - \text{prime, for all } i\}$$

Definition 1.1.18. The **Krull dimension of an A-module** M is defined to be

$$\dim(M) = \dim(A/\mathrm{Ann}(M)).$$

Definition 1.1.19. Let $A^{\oplus n} = \underbrace{A \oplus \cdots \oplus A}_{n-\text{copies}}$. An A-module M is **finitely generated** if there exists a surjective A-module morphism $A^{\oplus n} \to M$. Equivalently, M is finitely generated if there exist $m_1, \ldots, m_n$ such that for any $m \in M$, we can represent m as a sum

$$m = a_1 m_1 + a_2 m_2 + \cdots + a_n m_n$$

for some $a_1, \ldots, a_n \in A$. If M is finitely generated by $\{m_1, \ldots m_n\}$, and we also have that $a_1 m_1 + \cdots + a_n m_n = 0$ implies $a_i = 0$ for all i, then we say that $\{m_1, \ldots, m_n\}$ is a **basis** of M.

If A is a ring, then A is a module over itself, and moreover, admits a basis consisting of the unit element 1_A.

Definition 1.1.20. Let I be some indexing set and for each $i \in I$, let $A_i = A$, viewing each A_i as an A-module over itself. A **free module** is a module M that admits a basis, i.e., is of the form

$$M = \bigoplus_{i \in I} A_i$$

where a natural basis consists of elements e_i of M whose ith component is the identity element of A and all other components are zero.

Definition 1.1.21. The **rank** of a free A-module M is the cardinality of a basis of M.

Definition 1.1.22. An A-module is **finitely presented** if there exists $m, n \in \mathbb{Z}^+$ such that there is an exact sequence

$$A^{\oplus n} \longrightarrow A^{\oplus m} \longrightarrow M \longrightarrow 0$$

We call this right exact sequence a **presentation** of M.

Definition 1.1.23. A ring A is **noetherian** if every ideal of A is finitely generated. Similarly, an A-module M is **noetherian** if every submodule of M is finitely generated.

Proposition 1.1.24. *Let M be an A-module and let N be a submodule of M. Then M is noetherian if and only if both N and M/N are noetherian.*

Proof. Suppose M is noetherian and let N be a submodule of M. Since a submodule of a submodule is itself a submodule, by the noetherian property of M, any submodule of N is finitely generated, giving us that N is noetherian.

Let us now consider the quotient module M/N and let $\phi : M \to M/N$ be the surjection $m \mapsto m + N$. Let L be a submodule of M/N. Since the preimage of a submodule (under a module morphism) is a submodule of the domain, i.e., $\phi^{-1}(L) \subseteq M$ is a submodule, we get that $\phi^{-1}(L)$ is finitely generated. However, since ϕ is surjective, the finite number of generators of $\phi^{-1}(L)$ also generate all of the image of L under ϕ, giving us that L is finitely generated.

Conversely, suppose N and M/N are noetherian. Let L be a submodule of M and $\phi : M \to M/N$ be the canonical surjection. Then it follows that $L \cap N$ and $\text{im}(L)$ are finitely generated as submodules of N and M/N, respectively. Let $\{x_i\}_{i=1}^k \subseteq L$ be the finite set that generates $\text{im}(L)$ in M/N and let $\{y_j\}_{j=1}^r$ be the finite set of generators for $L \cap N$. Then for any $m \in L$, we get that

$$m = \sum_{i=1}^{k} a_i x_i + N \ \text{ in } M/N, \text{ where } a_i \in A$$

and since the $x_i \in L$ by assumption, we get that

$$m - \sum_{i=1}^{k} a_i x_i \in N \cap L.$$

Since the $\{y_j\}$ generate $N \cap L$, it follows that

$$m - \sum_{i=1}^{k} a_i x_i = \sum_{j=1}^{r} b_j y_j$$

and so we get that

$$m = \sum_{i=1}^{k} a_i x_i + \sum_{j=1}^{r} b_j y_j$$

giving us that L is finitely generated by the $\{x_i\}_{i=1}^{k}$ and $\{y_j\}_{j=1}^{r}$. □

Corollary 1.1.25. *Let A be a noetherian ring. For any positive integer $n \in \mathbb{Z}^+$, the free module $A^{\oplus n}$ is noetherian.*

Proof. We prove this by induction. For $n = 1$, we are done. Now assuming the statement holds for any $n \in \mathbb{N}$, consider the following short exact sequence

$$0 \longrightarrow A^{\oplus n} \longrightarrow A^{\oplus n+1} \longrightarrow A \longrightarrow 0.$$

By the induction hypothesis, we have that $A^{\oplus n}$ is noetherian and by Proposition 1.1.24, it follows that $A^{\oplus n+1}$ is noetherian. □

Corollary 1.1.26. *Let A be a noetherian ring and let M be finitely generated A-module. Then M is a noetherian A-module.*

Proof. Since M is finitely generated, we have the exact sequence $A^{\oplus n} \to M \to 0$. Thus, M is a quotient module of $A^{\oplus n}$, which we have shown is noetherian. Since the quotient module of a noetherian module is itself noetherian, we conclude that M is a noetherian A-module. □

Proposition 1.1.27. *If A is a noetherian ring, then every finitely generated A-module is finitely presented.*

Proof. Let M be a finitely generated A-module. Then there exists an exact sequence

$$A^{\oplus m} \xrightarrow{f} M \longrightarrow 0$$

for some $m \in \mathbb{Z}^+$. Since A is finitely generated, we get that $A^{\oplus m}$ is a finitely generated A-module and therefore ker(f) is finitely generated, giving rise to another exact sequence

$$A^{\oplus n} \longrightarrow \text{ker}(f) \longrightarrow 0$$

for some $n \in \mathbb{Z}^+$. Combining the two sequences gives us our desired result. □

1.2 Tensor Product

Definition 1.2.1. Let M, N, L be A-modules. A map $f : M \times N \to L$ is A-**bilinear** if f is A-linear in each variable, i.e., for all $m, m' \in M, n, n' \in N$, and $a \in A$, the following axioms are satisfied:

1. $f(m + m', n) = f(m, n) + f(m', n)$
2. $f(am, n) = af(m, n)$
3. $f(m, n + n') = f(m, n) + f(m, n')$
4. $f(m, am) = af(m, n)$

Even though M and N are A-modules, we note that $M \times N$ is simply a cartesian product and does not necessarily have an A-action defined on it and therefore $M \times N$ is not necessarily an A-module itself.

Example 1.2.2. Many operations in linear algebra are bilinear mappings. Let $\mathbb{k}$ be a field and let V be a $\mathbb{k}$-vector space.

1. Suppose $\mathbb{k} = \mathbb{R}$. Consider $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$, the (real) inner product of a $\mathbb{R}$-vector space. It should be noted that the complex inner product is not bilinear, but rather conjugate-linear in one of the variables.

2. Let $M_{2\times 2}(\mathbb{k})$ denote the space of 2×2 matrices over a field $\mathbb{k}$. There are only 4 entries, but we can treat the matrix column-wise, giving us that $M_{2\times 2}(\mathbb{k}) \cong \mathbb{k}^2 \times \mathbb{k}^2$ is an isomorphism at the level of sets. Giving this a vector space structure via the direct sum, we get
$$M_{2\times 2}(\mathbb{k}) \cong \mathbb{k}^2 \times \mathbb{k}^2 \xrightarrow{\det} \mathbb{k}$$
gives us a bilinear map.

Definition 1.2.3. Let M and N be A-modules. The **tensor product** of M and N over A is an A-module $M \otimes_A N$, together with a bilinear map $\otimes : M \times N \to M \otimes_A N$ satisfying the following universal property:

For every A-module L and every bilinear map $f : M \times N \to L$, there exists a unique A-module morphism $\varphi : M \otimes_A N \to L$ making the following diagram commute:

$$\begin{array}{ccc} M \times N & \xrightarrow{f} & L \\ {\scriptstyle \otimes}\downarrow & \nearrow_{\varphi} & \\ M \otimes_A N & & \end{array}$$

The tensor product does exist in the category of A-modules and is unique up to unique isomorphism. It is typically denoted as $M \otimes_A N$. For proof of existence and uniqueness, see [AM94] pages 24 and 25. We drop the subscript A if the base ring is clear.

Proposition 1.2.4. *Let A be a ring, and M, N, L be A-modules and let $\{M_i\}_{i \in I}$ be a family of A-modules. Then we have the following:*

(a) $M \otimes_A N \cong N \otimes_A M$.

(b) $M \otimes_A A \cong M$.

(c) $(L \otimes_A M) \otimes_A N \cong L \otimes_A (M \otimes_A N)$.

(d) $(\bigoplus_{i \in I} M_i) \otimes_A N \cong \bigoplus_{i \in I}(M_i \otimes_A N)$.

(e) For all A-module morphisms $f : M \to N$, there is an A-module morphism $f \otimes \mathrm{id}_L : M \otimes L \to N \otimes L$ given by $m \otimes \ell \longmapsto f(m) \otimes \ell$.

Proof. See [AM94] page 26 for details. □

Proposition 1.2.5 (Right-exactness of Tensors)**.** *Consider an exact sequence of A-modules*

$$N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0.$$

Then for any A-module M, the sequence

$$N' \otimes_A M \xrightarrow{f \otimes \mathrm{id}_M} N \otimes_A M \xrightarrow{g \otimes \mathrm{id}_M} N'' \otimes_A M \longrightarrow 0$$

is also exact.

Proof. See [AM94] pages 28 and 29 for details. □

Example 1.2.6. It is natural to ask if tensors are left exact as well. However, this fails generally at the level of rings, and therefore at the level of modules as well, particularly using Property (b) in Proposition 1.2.4. To see why, let n be a positive integer and consider the left exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{-\cdot n} \mathbb{Z}.$$

Tensoring this sequence with $\mathbb{Z}/n\mathbb{Z}$ over the base ring $\mathbb{Z}$, we get

$$\mathbb{Z}/n\mathbb{Z} \xrightarrow{-\cdot n} \mathbb{Z}/n\mathbb{Z}$$

which sends every element to zero, i.e., $\cdot n$ is the zero map and so left exactness is not necessarily preserved.

1.3 Free Resolutions and Flatness

Definition 1.3.1. A **free resolution** of an A-module M is an exact sequence

$$\cdots \longrightarrow F_\ell \xrightarrow{f_\ell} F_{\ell-1} \to \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0$$

where each F_ℓ is a free over A. If there exists a number $\ell \in \mathbb{N}$ such that $F_\ell \neq 0$ and $F_i = 0$ for all $i > \ell$, then we say that the free resolution is **finite with length** ℓ. If M has a finite free resolution, the minimal length among all finite free resolutions of M is called **projective dimension of** M and is denoted $\mathrm{pd}(M)$. If there is no free resolution of finite length, then $\mathrm{pd}(M) = \infty$.

By convention, in a free resolution, $F_i = 0$ for all $i < 0$.

As we saw, the tensor product generally fails to be left exact. We can measure how badly the tensor fails exactness using free resolutions and homology modules:

Let

$$\mathcal{F}.: \quad \cdots \longrightarrow F_{i+1} \xrightarrow{f_{i+1}} F_i \xrightarrow{f_i} F_{i-1} \longrightarrow \cdots$$

be a free resolution of an A-module N. Then for any A-module M, we can consider the following induced chain complex

$$M \otimes_A \mathcal{F}.: \quad \cdots \longrightarrow M \otimes_A F_{i+1} \xrightarrow{\mathrm{id}_M \otimes f_{i+1}} M \otimes_A F_i \xrightarrow{\mathrm{id}_M \otimes f_i} M \otimes_A F_{i-1} \longrightarrow \cdots .$$

Definition 1.3.2. The **Tor-modules** $\mathrm{Tor}_i^A(M, N)$ are defined to be

$$\mathrm{Tor}_i^A(M, N) = \ker(\mathrm{id}_M \otimes f_i)/(\mathrm{im}(\mathrm{id}_M \otimes f_{i+1})) = H_i(M \otimes_A \mathcal{F}.).$$

Thus, from this definition, we have that if $i < 0$, then $\mathrm{Tor}_i^A(M, N) = 0$ and moreover, if $i = 0$, then $\mathrm{Tor}_0^A(M, N) = M \otimes_A N$.

Definition 1.3.3. Let A be a ring and let M be an A-module. Let $C.$ be the sequence of A-modules and morphisms

$$C_{\cdot} : \quad \cdots \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow \cdots .$$

Consider the sequence

$$C. \otimes M : \quad \cdots \longrightarrow N' \otimes M \longrightarrow N \otimes M \longrightarrow N'' \otimes M \longrightarrow \cdots$$

We say that M is **flat over** A if for every exact sequence $C.$ the sequence $C. \otimes M$ is again exact. We say that M is **faithfully flat** if every sequence $C.$ is exact if and only if $C. \otimes M$ is exact.

Definition 1.3.4. If $f : A \to B$ is a ring morphism, and B is flat as an A-module, we say f is a **flat morphism** or B is a **flat A-algebra**.

Since flat A-modules preserve exactness of tensors, and Tor measures the failure of the tensor product to be exact, we have the following proposition:

Proposition 1.3.5. *An A-module M is flat if and only if* $\mathrm{Tor}_i^A(M, N) = 0$ *for every A-module N for all $i \neq 0$.*

Chapter 2

Syzygies

One method of better understanding an algebraic object is to decompose the object into smaller components. One way of decomposing objects is to introduce a *grading*. In this chapter, we take modules, module morphisms, and free resolutions from Chapter 1 and introduce their graded counterparts. In Section 2.2, we will also introduce new invariants called the Betti numbers, the graded Betti numbers, and the Castelnuovo-Mumford regularity, which will provide us more information about our modules and ideals.

2.1 Graded Rings and Modules

Definition 2.1.1. Let $(G, +)$ be an abelian monoid. A ring A is **G-graded** if it can be written in the form

$$A = \bigoplus_{g \in G} A_g,$$

where the A_g are abelian groups satisfying the multiplication property $A_g \cdot A_h \subseteq A_{g+h}$ for all $g, h \in G$. Similarly, an A-module M is a **G-graded A-module** if M has a decomposition

$$M = \bigoplus_{g \in G} M_g,$$

where the M_g are abelian groups satisfying the property $A_g M_h \subseteq M_{g+h}$ for all $g, h \in G$.

An ideal $I \subseteq A$ is a **graded ideal** (or homogeneous) if $I = \bigoplus_{g \in G} I \cap A_g$, i.e., I is graded as an A-module.

Example 2.1.2. Let $A = R[x_0, \dots, x_m]$ for some ring R. Denoting A_n to be the R-module generated by homogenous polynomials of degree n, i.e.,

$$A_n = \left\{p(x_0, \dots, x_m) : \text{degree of individual terms of } p(x_0, \dots x_m) = n\right\} \cup \{0\},$$

allows us to write

$$A = \bigoplus_{n \geq 0} A_n$$

giving us that A is $\mathbb{N}$-graded. If we allow $A_i = 0$ for all $i < 0$, then we have that A is $\mathbb{Z}$-graded as well.

Example 2.1.3. Let $\mathbb{k}$ be a field and consider the polynomial ring $\mathbb{k}[x, y]$. Under the standard grading introduced in Example 2.1.2, the ideal $(x^2 + y)$ is *not* homogenous, but the ideal $(x^2 + y^2, y^3)$ is homogenous.

Definition 2.1.4. Let $M = \oplus_{g \in G} M_g$ be a graded A-module and let $d \in G$ be fixed. Define

$$M(d) = \bigoplus_{g \in G} M(d)_g$$

with $M(d)_g := M_{d+g}$. Then $M(d)$ is a graded A-module and is called the d^{th} **twist** (or **shift**) of M.

Definition 2.1.5. Let M, N be graded A-modules and consider a morphism $f : M \to N$. We say f is **graded** or **homogeneous of degree** d if $f(M_n) \subseteq N_{d+n}$ for all $n \in G$. We say M and N are **isomorphic as graded** A**-modules** if there exists a homogeneous isomorphism between M and N.

Remark. If $f : M \to N$ is a morphism of degree d, then $f : M(-d) \to N$ is a morphism of degree 0.

Definition 2.1.6. A $\mathbb{Z}$-graded A-module M is said to be **free** if there is an isomorphism of graded A-modules

$$\phi : \bigoplus_{i \in I} A(n_i) \cong M, \quad n_i \in \mathbb{Z}.$$

2.2 Studying Syzygies

Lemma 2.2.1 (Nakayama)**.** *Let A be a $\mathbb{N}$-graded noetherian ring and let $\mathfrak{m}$ denote the homogenous maximal ideal of A, which is generated by forms of positive degree. Let M be finitely generated graded A-module and let $m_1, \ldots, m_n \in M$ generate $M/\mathfrak{m}M$. Then $m_1, \ldots, m_n$ generate M.*

Proof. For a discussion of the proof, see [AM94], pages 21 and 22. □

An example will be given after the few first definitions.

Definition 2.2.2. Let A be a noetherian ring and M be a graded A-module. A **graded free resolution** of M is a complex of free A-modules

$$\cdots \longrightarrow F_\ell \xrightarrow{f_\ell} F_{\ell-1} \to \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0$$

where each f_i is a degree 0 morphism and each F_i is free and of the form $F_i = A(-d_1) \oplus A(-d_2) \oplus \cdots \oplus A(-d_p)$ where $A(-d)$ denotes the degree d twisted component of A, i.e., $(A(d))_t = A_{d+t}$. If there exists a number $\ell \in \mathbb{N}$ such that $F_\ell \neq 0$ and $F_i = 0$ for all $i > \ell$, then we say that the graded free resolution is **finite with length** ℓ.

While constructing a graded free resolution of M is straightforward, we must be concerned whether *finite* graded free resolutions of M exists. The proof of existence was addressed and given by Hilbert for the case of polynomial rings over fields:

Theorem 2.2.3 (Hilbert's Syzygy Theorem)**.** *Let $\Bbbk$ be a field and let M be an A-module where A is the polynomial ring $A = \Bbbk[x_1, \ldots, x_n]$. If M is finitely generated, then M has a finite graded resolution of length at most n, the number of variables in A.*

Proof. See [Eis05], Section 2B on pages 20 and 21. □

Example 2.2.4. Let $A = \Bbbk[x, y]$ and let $I = (x^2, xy, y^3)$. Letting M be the quotient ring A/I and viewing M as an A-module, the minimal resolution of M is given by

$$0 \leftarrow M \leftarrow A \xleftarrow{\begin{bmatrix} x^2 & xy & y^3 \end{bmatrix}} A(-2)^2 \oplus A(-3) \xleftarrow{\begin{bmatrix} y & 0 \\ -x & y^2 \\ 0 & -x \end{bmatrix}} A(-3) \oplus A(-4) \leftarrow 0.$$

We note that the length of this resolution is 2.

Definition 2.2.5. A graded free resolution

$$\cdots \longrightarrow F_\ell \xrightarrow{f_\ell} F_{\ell-1} \to \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0$$

is said to be **minimal** if for each i, we have that $\text{im}(f_i) \subseteq \mathfrak{m}F_{i-1}$ where $\mathfrak{m}$ denotes the homogenous maximal ideal of the base ring of M.

Remark. Despite the discussion so far, it is not yet clear whether minimal graded free resolutions exist for a given module, even for the case of a finitely generated A-module M where $A = \Bbbk[x_1, \ldots, x_n]$ as the free resolution given by Hilbert's Syzygy Theorem is not necessarily minimal. Thus, we trim a given free resolution until it becomes minimal, justified by Theorem 2.2.6, and this can be done by selecting a minimal system of generators.

To see why we can pick a minimal system of generators, let M be a module over some ring A and let

$$\cdots \longrightarrow F_\ell \xrightarrow{f_\ell} F_{\ell-1} \to \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0$$

be a graded free resolution of M. We can consider the right exact sequence

$$F_i \xrightarrow{f_i} F_{i-1} \longrightarrow \operatorname{im}(f_{i-1}) \longrightarrow 0$$

and the induced right exact sequence

$$F_i/\mathfrak{m}F_i \xrightarrow{g_i} F_{i-1}/\mathfrak{m}F_{i-1} \xrightarrow{\varphi} \operatorname{im}(f_{i-1})/\mathfrak{m}(\operatorname{im}(f_{i-1})).$$

The former right exact sequence is minimal if and only if $f_i(F_i) \subseteq \mathfrak{m}F_{i-1}$, which is equivalent to saying the map g_i in the induced sequence is the zero map and by exactness, this is true if and only if φ is an isomorphism, i.e., $F_{i-1}/\mathfrak{m}F_{i-1} \cong \operatorname{im}(f_{i-1})/\mathfrak{m}(\operatorname{im}(f_{i-1}))$. By Nakayama's Lemma, the generators of $F_{i-1}/\mathfrak{m}F_{i-1}$ are also generators of F_{i-1} and so φ is an isomorphism if and only if F_{i-1} maps to a minimal set of generators of $\operatorname{im}(f_{i-1})$. Thus, without any loss of generality, we can assume all free resolutions are minimal unless otherwise stated.

Noting that the construction of a minimal free resolution is artificial and dependent on choices, we surprisingly get that all minimal free resolutions of a given module are isomorphic to each other.

Theorem 2.2.6. *Let M be a finitely generated A-module. If $\mathcal{F}$ and $\mathcal{G}$ are minimal graded free resolutions of M, then there is a graded isomorphism of complexes $\mathcal{F} \to \mathcal{G}$ inducing the identity map on M. Moreover, any free resolution of M contains the minimal free resolution as a direct sum.*

Proof. See Theorem 20.2 on page 491 in [Eis95]. □

Example 2.2.7. Let $A = \Bbbk[x, y]$ and $I = (x^4, x^2y, y^2)$ be an ideal of A. Let M be the quotient ring A/I viewed as an A-module. A graded free resolution of M is given by

$$0 \leftarrow M \leftarrow A \xleftarrow{X} A(\text{-}4) \oplus A(\text{-}3) \oplus A(\text{-}2) \xleftarrow{Y} A(\text{-}5) \oplus A(\text{-}6) \oplus A(\text{-}4) \xleftarrow{Z} A(-6)^3 \leftarrow 0$$

where

$$X = \begin{bmatrix} x^4 & x^2y & y^2 \end{bmatrix},\ Y = \begin{bmatrix} 0 & -x^4 & -x^2 \\ -x^2 & 0 & y \\ y & y^2 & 0 \end{bmatrix},\ Z = \begin{bmatrix} y & y & y \\ -1 & -1 & -1 \\ x^2 & x^2 & x^2 \end{bmatrix}.$$

We note that this graded free resolution is not minimal, as the third mapping has elements not in the maximal ideal. Moreover, in the second mapping, we have a linear combination between the columns:

$$\begin{bmatrix} -x^4 \\ 0 \\ y^2 \end{bmatrix} = y \begin{bmatrix} 0 \\ -x^2 \\ y \end{bmatrix} + x^2 \begin{bmatrix} -x^2 \\ y \\ 0 \end{bmatrix}.$$

Thus to obtain the minimal graded free resolution, we trim the given resolution and obtain that the minimal resolution is given by

$$0 \leftarrow M \leftarrow A \xleftarrow{\begin{bmatrix} x^4 & x^2y & y^2 \end{bmatrix}} A(-2)\oplus A(-3)\oplus A(-4) \xleftarrow{\begin{bmatrix} 0 & -x^2 \\ -x^2 & y \\ y & 0 \end{bmatrix}} A(-4)\oplus A(-5) \leftarrow 0.$$

Definition 2.2.8. Consider a minimal free resolution of M over A:

$$\cdots \longrightarrow F_\ell \xrightarrow{f_\ell} F_{\ell-1} \to \cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0.$$

The i^{th} **syzygy module** of M is $\Omega^i(M) = \text{im}(f_{i+1}) = \ker(f_i)$. The rank of the i^{th} syzygy module is called the i^{th} **Betti number** and is denoted as β_i. Writing $F_i = \bigoplus A(-j)$, then a graded minimal free resolution of M is of the form

$$\cdots \to \bigoplus_j A(-j)^{\beta_{\ell,j}} \to \cdots \to \bigoplus_j A(-j)^{\beta_{1,j}} \to \bigoplus_j A(-j)^{\beta_{0,j}} \to M \to 0.$$

The exponents $\beta_{i,j}$ of the shifted modules $A(-j)$ are called the **graded Betti numbers** of M over A.

Example 2.2.9. Consider the previous example of minimal graded free resolution of M where $M = \mathbb{k}[x, y]/(x^4, x^2y, y^2)$ given by

$$0 \leftarrow M \leftarrow A \leftarrow A(-2) \oplus A(-3) \oplus A(-4) \leftarrow A(-4) \oplus A(-5) \leftarrow 0.$$

The 0th Betti number is 1, the 1st Betti number is 3, and the 2nd Betti number is 2. The graded Betti numbers are $\beta_{0,0} = 1; \beta_{1,2} = 1; \beta_{1,3} = 1; \beta_{1,4} = 1; \beta_{2,4} = 1; \beta_{2,5} = 1$.

To keep track of all of the graded Betti numbers, it is convenient to display these in a table. While it is ideal to place $\beta_{i,j}$ in the ith column and jth row, we note that all $\beta_{ij} = 0$ for $j < i$. Letting r denote the largest j in the minimal graded free resolution, we introduce the **Betti diagram** of M to be

$$\beta(M) = \begin{bmatrix} \beta_{00} & \beta_{11} & \beta_{22} & \cdots & \beta_{nn} \\ \beta_{01} & \beta_{12} & \beta_{23} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \beta_{0r} & \beta_{1,1+r} & \cdots & \cdots & \beta_{n,n+r} \end{bmatrix}$$

and so the Betti number $\beta_{i,j}$ is placed in the i^{th} column and $(j-i)^{th}$ row. So the Betti diagram of Example 2.2.9 would be:

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}.$$

The i^{th} Betti number can be retrieved by summing all entries in the i^{th} column. Using these Betti tables, we can identify the projective dimension of M since if

$$0 \longrightarrow F_\ell \xrightarrow{f_\ell} F_{\ell-1} \to \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0$$

is a finite minimal free graded resolution of M, then the definition of projective dimension in Definition 1.3.1 gives us

$$\text{pd}(M) = \sup\{i : F_i \neq 0\} = \sup\left\{i : \beta_{ij}(M) \neq 0 \text{ for some } j\right\}.$$

Definition 2.2.10. The **Castelnuovo-Mumford regularity** (or simply **regularity**) of a module is defined to be

$$\text{reg}_A(M) = \sup\left\{j - i : \beta_{ij}(M) \neq 0\right\}$$

Thus, we can read the regularity of a module by looking at the index of the final nonzero row of its graded Betti table.

So if M is the module in Example 2.2.9, we calculate that $\mathrm{pd}(M) = 2$ and $\mathrm{reg}_A(M) = 3$.

Although our concern about the choices made in the construction of a minimal graded free resolution has been addressed in Theorem 2.2.6, we include a verification that the graded Betti numbers are independent of these choices.

Proposition 2.2.11. *If $F.$ is the minimal free resolution of a finitely generated A-module M, and if $\Bbbk$ is the residue field $A/\mathfrak{m}$, then any minimal set of homogenous generators of F_i contains exactly $\dim_{\Bbbk}(\mathrm{Tor}_i^A(\Bbbk, M)_j)$ generators of degree j. That is, $\beta_{ij} = \dim_{\Bbbk} \mathrm{Tor}_i^A(\Bbbk, M)_j$.*

Proof. Let $\mathcal{F}. : \cdots \to F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0$ be a minimal free resolution of M. From this, it follows that the complex $\Bbbk \otimes_A \mathcal{F}.$ is

$$\Bbbk \otimes \mathcal{F}. : \cdots \to \Bbbk \otimes_A F_i \longrightarrow \cdots \longrightarrow \Bbbk \otimes_A F_1 \longrightarrow \Bbbk \otimes_A F_0 \longrightarrow \Bbbk \otimes_A M \longrightarrow 0.$$

However, since each F_i is a free module, it is isomorphic to b_i copies of A and since $\Bbbk \otimes_A A \cong \Bbbk$, we get that the complex $\Bbbk \otimes \mathcal{F}.$ is isomorphic to

$$\cdots \longrightarrow \Bbbk^{\oplus b_i} \longrightarrow \cdots \longrightarrow \Bbbk^{\oplus b_1} \longrightarrow \Bbbk^{\oplus b_0} \longrightarrow \Bbbk \otimes_A M \longrightarrow 0.$$

However, by the minimality of $\mathcal{F}.$, we get that all of the differentials of $\Bbbk \otimes \mathcal{F}.$ are zero maps and so we compute that $\mathrm{Tor}_i(\Bbbk, M) = \Bbbk \otimes F_i$. Thus, $\dim_{\Bbbk}(\mathrm{Tor}_i(\Bbbk, M)_j) = b_{ij}$ but by Nakayama's lemma, we conclude that $\beta_{ij} = b_{ij}$. □

Chapter 3

Rees Algebras and Regularity

The projective dimension, defined in Definition 1.3.1, and the Castelnuovo-Mumford regularity, introduced in Definition 2.2.10, are important tools in modern algebraic geometry and commutative algebra, used to measure the complexity of a given module M. One difference is that regularity also takes into account the degrees of the generators of M. Computing the projective dimension and regularity can be quite difficult, requiring the computation of syzygies and Gröbner bases.

We restrict our focus to homogeneous ideals. Given a homogenous ideal I in a polynomial ring $A = \mathbb{k}[x_1, \ldots, x_n]$, our main goal for this chapter is to study the complexity of integral powers of I. To do this, we first define the Rees algebra of I, an algebraic object that captures higher powers of the ideal, in Section 3.1. Studying the Rees algebra and taking its minimal free resolution then gives us information about the regularity and projective dimension of I^d for any $d \in \mathbb{N}$. In Section 3.2, we go through a detailed example on how to extract regularity and projective dimension from the Rees algebra of an ideal and in Section 3.3., we provide a bound on the regularity of I^d using the Rees algebra.

3.1 Rees Algebras

Definition 3.1.1. Let I be an ideal in A. Let $\mathcal{I} = \{I^n\}$ be the I-adic filtration of A. The **Rees algebra** $\mathcal{R}(\mathcal{I})$ is defined to be

$$\mathcal{R}(\mathcal{I}) = \bigoplus_{n \in \mathbb{N}} I^n t^n = A \oplus It \oplus \cdots \oplus I^m t^m \oplus \cdots = A[It] \subseteq A[t],$$

where t is an indeterminate over A. Intuitively, the Rees algebra is an algebraic object which captures all non-negative powers of an ideal I.

Specializing to the case that $A = \Bbbk[x_1, \dots, x_n]$, the Rees algebra of a homogenous ideal $I = (f_0, \dots, f_k)$ carries a natural bigraded structure. To see the bigrading, let $B = \Bbbk[x_1, \dots, x_n, w_0, \dots w_k] = A[w_0, \dots, w_k]$ be another polynomial ring. Define a ring morphism

$$\begin{aligned} B &\longrightarrow A[t] = \Bbbk[x_1, \dots, x_n, t] \\ x_i &\longmapsto x_i \\ w_j &\longmapsto f_j t \end{aligned}$$

and so we get that $\deg(x_i) = (1, 0)$ and $\deg(w_i) = (\deg(f_i), 1)$.

Example 3.1.2. Let $I = (x^4, x^2y, y^2)$ be an ideal of $R = \Bbbk[x, y]$. Denoting the Rees algebra of I to be $R[It]$, we get that $R[It]$ is the quotient of the bigraded ring $S = R[u, v, w]$ where the degrees of the variables of S are given by

$$\begin{aligned} &\deg(x) = \deg(y) = (1, 0); \\ &\deg(u) = (4, 1); \quad \deg(v) = (3, 1); \quad \deg(w) = (2, 1). \end{aligned}$$

From this, we can explicitly compute the Rees algebra of I to be

$$R[It] = R[u, v, w]/(v^2 - uw, x^2v - yu, yv - x^2w).$$

3.2 Resolutions of a Rees Algebras

Before diving into the general case, we first work out an example where we take free resolutions of the Rees algebra of an ideal I to obtain the minimal free resolution of I^d for large enough d. From this, we can easily read the regularity of I^d as well.

Taking Example 3.1.2 to be the set up, we compute a bigraded minimal free resolution of $R[It]$ over S, which gives us

$$0 \leftarrow R[It] \leftarrow S \xleftarrow{A} F_1 \xleftarrow{B} F_2 \leftarrow 0$$

where $F_1 = S(\text{-}6, \text{-}2) \oplus S(\text{-}5, \text{-}1) \oplus S(\text{-}4, \text{-}1)$, $F_2 = S(\text{-}8, \text{-}2) \oplus S(\text{-}7, \text{-}2)$ and the mappings

A and B are given by

$$A = \begin{bmatrix} v^2 - uw & x^2v - yu & yv - x^2w \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} x^2 & y \\ -v & -w \\ -u & -v \end{bmatrix}.$$

Picking any power d in the second degree allows us to read off free resolutions (possibly non-minimal) of I^d. For example, if we pick the $(*, 2)$-strand from this resolution, we obtain the free resolution

$$0 \leftarrow R[It]_{(*,2)} \leftarrow S_{(*,2)} \leftarrow F_{1,(*,2)} \leftarrow F_{2,(*,2)} \leftarrow 0.$$

Since we are restricting to the $(*, 2)$ degree in the bigrading, we follow the definition to compute that $R[It]_{(*,2)} = I^2t^2$. Moreover, we can compute that

$$S_{(*,2)} = Ru^2 \oplus Ruv \oplus Ruw \oplus Rv^2 \oplus Rvw \oplus Rw^2.$$

Introducing the variables u', v', w' to be placeholders for the u, v, w respectively, but with the change in degrees such that

$$\deg(u') = \deg(v') = \deg(w') = (0, 1),$$

we can reintroduce the twists to compute that

$$S_{(*,2)} = R(-8)u'^2 \oplus R(-7)u'v' \oplus R(-6)u'w' \oplus R(-6)v'^2 \oplus R(-5)v'w' \oplus R(-4)w'^2.$$

Similarly, since $F_{1,(*,2)} = S(-6, -2)_{(*,2)} \oplus S(-5, -1)_{(*,2)} \oplus S(-4, -1)_{(*,2)}$ and $F_{2,(*,2)} = S(-8, -2)_{(*,2)} \oplus S(-7, -2)_{(*,2)}$, compute for each component,

$$\begin{aligned} S(-6, -2)_{(*,2)} &= S_{(*-6,0)} = R(-6) \\ S(-5, -1)_{(*,2)} &= S_{(*-5,1)} = R(-5)u \oplus R(-5)v \oplus R(-5)w \\ &= R(-9)u' \oplus R(-8)v' \oplus R(-7)w' \\ S(-4, -1)_{(*,2)} &= S_{(*-4,1)} = R(-4)u \oplus R(-4)v \oplus R(-4)w \\ &= R(-8)u' \oplus R(-7)v' \oplus R(-6)w' \\ S(-8, -2)_{(*,2)} &= S_{(*-8,0)} = R(-8) \\ S(-7, -2)_{(*,2)} &= S_{(*-7,0)} = R(-7) \end{aligned}$$

to give us that

$$\begin{aligned} F_{1,(*,2)} &= S(-6,-2)_{(*,2)} \oplus S(-5,-1)_{(*,2)} \oplus S(-4,-1)_{(*,2)} \\ &= R(-6) \oplus R(-9)u' \oplus R(-8)v' \oplus R(-7)w' \oplus R(-8)u' \oplus R(-7)v' \oplus R(-6)w' \\ F_{2,(*,2)} &= S(-8,-2)_{(*,2)} \oplus S(-7,-2)_{(*,2)} \\ &= R(-8) \oplus R(-7). \end{aligned}$$

It may be tempting to conclude that the Betti table might look something like

$$\begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

but we note that this cannot be an actual Betti table for any module or ideal, since our resolution is non-minimal. If in the case that our resolution was said to be minimal with this given table, then our resolution fails exactness. We also see that $\operatorname{reg}(I^2) \leq 8$.

Analyzing the degree 0 graded morphism

$$S_{(*,2)} \xleftarrow{\begin{bmatrix} v^2 - uw & x^2v - yu & yv - x^2w \end{bmatrix}} F_{1,(*,2)}$$

we get

$$\begin{array}{c c} & \begin{array}{ccccccc} R(-6) & R(-9)u' & R(-8)v' & R(-7)w' & R(-8)u' & R(-7)v' & R(-6)w' \end{array} \\ \begin{array}{c} R(-8)u'^2 \\ R(-7)u'v' \\ R(-6)u'w' \\ R(-6)v'^2 \\ R(-5)v'w' \\ R(-4)w'^2 \end{array} & \begin{bmatrix} 0 & -y & 0 & 0 & 0 & 0 & 0 \\ 0 & x^2 & -y & 0 & y & 0 & 0 \\ -1 & 0 & 0 & -y & -x^2 & 0 & 0 \\ 1 & 0 & x^2 & 0 & 0 & y & 0 \\ 0 & 0 & 0 & x^2 & 0 & -x^2 & y \\ 0 & 0 & 0 & 0 & 0 & 0 & -x^2 \end{bmatrix} \end{array}.$$

We will call matrix above G, as it is a ginormous matrix.

Similarly, if we analyze the map

$$F_{1,(*,2)} \xleftarrow{\begin{bmatrix} x^2 & y \\ -v & -w \\ -u & -v \end{bmatrix}} F_{2,(*,2)}$$

we compute that

$$\begin{array}{c} \\ R(-6) \\ R(-9)u' \\ R(-8)v' \\ R(-7)w' \\ R(-8)u' \\ R(-7)v' \\ R(-6)w' \end{array} \begin{array}{c} \begin{array}{cc} R(-8) & R(-7) \end{array} \\ \begin{bmatrix} x^2 & y \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \end{array}.$$

Call this second matrix H.

We begin trimming our free resolution by taking G and begin using row and column operations on G. Denoting the columns of G as $C_1, \ldots, C_7$ and the rows of G as $R_1, \ldots R_6$, perform the following operations:

1. $R_3 \to R_3 + R_4$; this is the new 3rd row;
2. $C_6 \to C_4 + C_6$; this is the new 4th column;
3. $C_3 \to C_3 + C_5$;
4. $C_3 \to C_3 - x^2 C_1$;
5. $C_6 \to C_6 - yC_1$;
6. Swap $R_4 \leftrightarrow R_1$.

Recording these row and column operations as elementary matrices, respectively, gives us

$$E_G = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad C_G = \begin{bmatrix} 1 & 0 & -x^2 & 0 & 0 & -y & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

such that

$$E_GGC_G = \begin{array}{c} \\ R(-6) \\ R(-7) \\ R(-6) \\ R(-8) \\ R(-5) \\ R(-4) \end{array} \begin{array}{c} \begin{array}{ccccccc} R(-6) & R(-9) & R(-8) & R(-7) & R(-8) & R(-7) & R(-6) \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x^2 & 0 & 0 & y & 0 & 0 \\ 0 & 0 & 0 & -y & -x^2 & 0 & 0 \\ 0 & -y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x^2 & 0 & 0 & y \\ 0 & 0 & 0 & 0 & 0 & 0 & -x^2 \end{bmatrix} \end{array}.$$

We note that this concretely informs us that our resolution of I^2 is non-minimal as there are entries that are not in the maximal ideal. Moreover, as we have composed G with E_G and precomposed it with C_G, we have that $E_GGC_G : C_G^{-1}F_{1,(*,2)} \to E_GS_{(*,2)}$ and in order to preserve exactness, we must change our bases accordingly. Diagrammatically, we have that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_{2,(*,2)} & \xrightarrow{H} & F_{1,(*,2)} & \xrightarrow{G} & S_{(*,2)} & \longrightarrow & I^2 & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle C_G^{-1}} & & \downarrow{\scriptstyle E_G} & & & & \\ 0 & \longrightarrow & F_{2,(*,2)} & \xrightarrow{C_G^{-1}H} & C_G^{-1}F_{1,(*,2)} & \xrightarrow{E_GGC_G} & E_GS_{(*,2)} & \longrightarrow & I^2 & \longrightarrow & 0 \end{array}$$

This reduction does reveal that under a certain choice of bases, the degree 0 morphism G is of the form

$$\begin{aligned} R(-6) \oplus \big(R(-9) \oplus R(-8)^2 \oplus R(-7)^2 \oplus R(-6)\big) &\longrightarrow R(-6) \oplus \left(\bigoplus_{j=4}^{8} R(-j)\right) \\ (v, w) &\longmapsto (v, \widetilde{G}w) \end{aligned}$$

where $w \in R(-9) \oplus R(-8)^2 \oplus R(-7)^2 \oplus R(-6)$ and $v \in R(-6)$ and $\widetilde{G}$ is the lower right block matrix in the matrix EGC.

We will also begin the same procedure fo H. Recall that

$$H = \begin{array}{r} \\ R(-6) \\ R(-9)u \\ R(-8)v \\ R(-7)w \\ R(-8)u \\ R(-7)v \\ R(-6)w \end{array} \begin{array}{c} \begin{array}{cc} R(-8) & R(-7) \end{array} \\ \begin{bmatrix} x^2 & y \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \end{array}.$$

and to preserve exactness under the change of basis, we compose H with C_G^{-1}; computing this change gives us

$$\widetilde{H} := C_G^{-1}H = \begin{bmatrix} 1 & 0 & x^2 & 0 & 0 & y & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^2 & y \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

Given this, we can begin performing more elementary operations on $\widetilde{H}$ and recording these into another elementary matrices gives us

$$E_{\widetilde{H}} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad C_{\widetilde{H}} = I_2.$$

Using these matrices, we compute that

$$E_{\widetilde{H}}\widetilde{H} = \begin{array}{c} \\ R(-8) \\ R(-7) \\ R(-6) \\ R(-7) \\ R(-8) \\ R(-9) \\ R(-6) \end{array} \begin{array}{c} \begin{array}{cc} R(-8) & R(-7) \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}.$$

Again we remark that this morphism is also non-minimal and that

$$E_{\widetilde{H}}\widetilde{H} : F_{2,(*,2)} \longrightarrow E_{\widetilde{H}}C_G^{-1}F_{1,(*,2)}.$$

Using the following commutative diagram to keep track of all the changes,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_{2,(*,2)} & \xrightarrow{H} & F_{1,(*,2)} & \xrightarrow{G} & S_{(*,2)} & \longrightarrow & I^2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & {\scriptstyle C_G^{-1}}\downarrow & & \downarrow{\scriptstyle E_G} & & & & \\
0 & \longrightarrow & F_{2,(*,2)} & \xrightarrow{C_G^{-1}H} & C_G^{-1}F_{1,(*,2)} & \xrightarrow{E_GGC_G} & E_GS_{(*,2)} & \longrightarrow & I^2 & \longrightarrow & 0 \\
 & & {\scriptstyle C_{\widetilde{H}}^{-1}=\mathrm{id}}\downarrow & & \downarrow{\scriptstyle E_{\widetilde{H}}} & & \downarrow{\scriptstyle \mathrm{id}} & & & & \\
0 & \longrightarrow & F_{2,(*,2)} & \xrightarrow{E_{\widetilde{H}}C_G^{-1}H} & E_{\widetilde{H}}C_G^{-1}F_{1,(*,2)} & \xrightarrow{E_GGC_GE_{\widetilde{H}}^{-1}} & E_GS_{(*,2)} & \longrightarrow & I^2 & \longrightarrow & 0
\end{array}$$

we get that under this change of basis that the morphism $E_{\widetilde{H}}C_G^{-1}H$ acts as the identity map on a copy of $R(-8) \oplus R(-7) \to R(-8) \oplus R(-7)$. Thus, we have that

$$R(-8) \oplus R(-7) \oplus 0 \xrightarrow{(\mathrm{id},\mathrm{id},0)} R(-8) \oplus R(-7) \oplus R^5$$

where $R^5 = R(-6)^2 \oplus R(-7) \oplus R(-8) \oplus R(-9)$.

Thus, we get that under a specific choice of bases, we get that the graded free resolution of I^2 is

$$\begin{array}{ccccccccccc} & & & & R(-6) & \xleftarrow[\text{id}]{} & R(-6) & & & & \\ & & & & \oplus & & \oplus & & & & \\ 0 & \longleftarrow & I^2 & \longleftarrow & \bigoplus_{j=4}^{8} R(-j) & \longleftarrow & \bigoplus_{i=6}^{9} R(-i) & \longleftarrow & 0 & \longleftarrow & 0 \\ & & & & & & \oplus & & \oplus & & \\ & & & & & & R(-7) \oplus R(-8) & \xleftarrow[\text{id}]{} & R(-7) \oplus R(-8). & & \end{array}$$

We can now obtain minimality by trimming the resolution of spaces which get mapped onto itself by the identity map and we get that the minimal graded free resolution of I^2 is now given by

$$0 \longleftarrow I^2 \longleftarrow \bigoplus_{j=4}^{8} R(-j) \longleftarrow \bigoplus_{i=6}^{9} R(-i) \longleftarrow 0,$$

and therefore the graded Betti table for I^2 is given by

$$\begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \end{array}.$$

From this, we conclude that $\text{reg}(I^2) = 8$ and $\text{pd}(I^2) = 2$.

3.3 A Bound on Regularity via Rees Algebras

Using the notation in Section 3.1, we are now ready to provide a bound on the regularity of I^d. Whieldon [Whi14] proved the same result in the case that the generators of I are all of the same degree.

Theorem 3.3.1. *Let $A = \Bbbk[x_1, \dots, x_n]$, $d \in \mathbb{Z}^+$ and $I = (f_0, \dots, f_k)$ be a homogenous ideal in A. Let $B = A[w_0, \dots, w_k]$ with a $\mathbb{Z}^2$-bigrading of B given by* $\text{bideg}(x_i) = (1, 0)$ *and* $\text{bideg}(w_i) = (\deg(f_i), 1)$*, i.e., $B = \bigoplus_{j,m} B(-j, -m)$. We have an upper bound on* *the regularity of I^d given by*

$$\operatorname{reg}(I^d) \leq \max_{0\leq \ell \leq k} \left\{ j + \sum_{i=0}^{k} a_i \deg(f_i) - \ell : \beta_{\ell,j,m} \neq 0, \sum_{i=0}^{k} a_i = d - m \right\}.$$

Proof. Denoting $A[It]$ to be the Rees algebra of I, and viewing it as a B-module, by Hilbert's Syzygy theorem, we get a bigraded free resolution, which we can take to be minimal, given by

$$0 \leftarrow A[It] \xleftarrow{\phi_0} \bigoplus_{j,m} B(-j,-m)^{\beta_{0,j,m}} \leftarrow \cdots \xleftarrow{\phi_k} \bigoplus_{j,m} B(-j,-m)^{\beta_{k,j,m}} \leftarrow 0.$$

Restricting to the $(*, d)$ degree in the bigrading, we obtain a graded free resolution of A-modules, since a homogenous strand of an exact complex is itself exact,

$$0 \leftarrow A[It]_{(*,d)} \leftarrow \bigoplus_{j,m} B(-j,-m)^{\beta_{0,j,m}}_{(*,d)} \leftarrow \cdots \leftarrow \bigoplus_{j,m} B(-j,-m)^{\beta_{k,j,m}}_{(*,d)} \leftarrow 0.$$

However, we note that this graded free resolution of A-modules is not necessarily minimal.

At each free module, using the bigraded structure of the Rees algebra, we can compute that

$$B(-j,-m)_{(*,d)} = \bigoplus_{\substack{\mathbf{a}\in\mathcal{P}_{k+1}(d-m)\\ \mathbf{a}=(a_0,\ldots,a_k)}} A(-j) w_0^{a_0} w_1^{a_1} \cdots w_k^{a_k}$$

where $\mathbf{a} = (a_0, \ldots, a_k) \in \mathbb{N}^{k+1}$ and $\mathcal{P}_{k+1}(N)$ denotes all possible partitions of $N \in \mathbb{N}$ into $k+1$ non-negative integers. Since each w_i carries a $\operatorname{bideg}(w_i) = (\deg(f_i), 1)$, we can introduce placeholder variables v_i such that $\operatorname{bideg}(v_i) = (0, 1)$. From this, it follows that each

$$A(-j) w_0^{a_0} w_1^{a_1} \cdots w_k^{a_k} = A(-j - \sum_{i=0}^{k} a_i \deg(f_i)) v_0^{a_0} \cdots v_k^{a_k}.$$

Since the v_i are simply placeholders, our (possibly non-minimal) free resolution of I^d is given by

$$0 \leftarrow I^d \leftarrow \bigoplus_{j,m} \bigoplus_{\substack{\mathbf{a}\in\mathcal{P}_{k+1}(d-m)\\ \mathbf{a}=(a_0,\ldots,a_k)}} A(-j - \sum_{i=0}^{k} a_i \deg(f_i))^{\beta_{0,j,m}} \leftarrow \cdots$$

$$\cdots \leftarrow \bigoplus_{j,m} \bigoplus_{\substack{\mathbf{a}\in\mathcal{P}_{k+1}(d-m)\\ \mathbf{a}=(a_0,\ldots,a_k)}} A(-j - \sum_{i=0}^{k} a_i \deg(f_i))^{\beta_{k,j,m}} \leftarrow 0.$$

Thus, we have an upper bound on $\operatorname{reg}(I^d)$ given by

$$\operatorname{reg}(I^d) \leq \max_{0 \leq \ell \leq k} \left\{ j + \sum_{i=0}^{k} a_i \deg(f_i) - \ell : \beta_{\ell,j,m} \neq 0, \sum_{i=0}^{k} a_i = d - m \right\}.$$

□

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